LOYOLA COLLEGE (AUTONOMOUS) CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2025**

PMT2MC02 – REAL ANALYSIS-II

Date: 26-04-2025 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

	SECTION A – K1 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
1	**Answer the following**
a)	Test the existence of the simultaneous limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}, x^2 + y^2 \neq 0$.
b)	What do you mean by approximating measure for a set A?
c)	Obtain the positive part of the function f defined on $[-3,3]$ by $f(x) = x^2 - 4$.
d)	Give an example of a measure space.
e)	Define a measurable rectangle in a product measurable space.
	SECTION A – K2 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
2	**MCQ**
a)	The directional derivative of a function $f(x,y)$ in the direction of a unit vector u is (i) $(\nabla f).(u)$ (ii) $D_1 f$ (iii) $D_2 f$ (iv) $D_{12} f$
b)	Which of the following sets has Lebesgue outer measure zero? (i) Set of rational numbers in [0, 1] (ii) The interval (0, 1) (iii) Set of irrational numbers in [0, 1] (iv) The real line R.
c)	The Lebesgue Dominated Convergence Theorem requires that: (i) $lim f_n = f\ a.e.$ (ii) There exists an integrable function g such that $\lvert f_n \rvert \leq g$ (iii) f_n is bounded (iv) $lim f_n = f\ a.e.$ and there exists an integrable function g such that $\lvert f_n \rvert \leq g$.
d)	Which of the following functions satisfies the Jensen's inequality $\psi(\int f d\mu) \leq \int \psi \circ f d\mu$? (i) $f(x) = x^2$ (ii) $f(x) = -x$ (iii) $f(x) = \log x$ (iv) $f(x) = sinx$.
e)	The product measure of two σ-finite measures is always (i) Finite (ii) σ-finite (iii) Infinite (iv) Countable
	SECTION B – K3 (CO2)
	Answer any THREE of the following **(3 x 10 = 30)**
3	Examine the following statements: (i) $\lVert \boldsymbol{A} \rVert < \infty$ and $\boldsymbol{A}$ is a uniformly continuous map from R^n into R^m for $\boldsymbol{A} \in L(R^n, R^m)$, (ii) When $\boldsymbol{A}, \boldsymbol{B} \in L(R^n, R^m)$, $\lVert \boldsymbol{A} + \boldsymbol{B} \rVert \leq \lVert \boldsymbol{A} \rVert + \lVert \boldsymbol{B} \rVert$ and $\lVert c\boldsymbol{A} \rVert = \lvert c \rvert \lVert \boldsymbol{A} \rVert$ where c is a scalar. (iii) If $\boldsymbol{A} \in L(R^n, R^m)\ and\ \boldsymbol{B} \in L(R^m, R^k)$, then $\lVert \boldsymbol{BA} \rVert \leq \lVert \boldsymbol{B} \rVert \lVert \boldsymbol{A} \rVert$.

4	Confirm the statement: Every interval is measurable.
5	Verify whether the Riemann integral of bounded function f over the finite interval $[a, b]$ is equal to the corresponding integral in the Lebesgue sense.
6	Define a measure μ on a $\sigma - ring\ \mathcal{S}$ and $\bar{\mathcal{S}} = \{E \,\Delta\, N : E, M \in \mathcal{S}\ and\ N \subseteq M\ with\ \mu(M) = 0\}$. Demonstrate that $\bar{\mathcal{S}}$ is a σ-ring.
7	Let $[\![X, \mathcal{S}, \mu]\!]$ and $[\![Y, \mathcal{T}, \nu]\!]$ be σ-finite measure spaces and f is integrable with respect to the product measure $\mu \times \nu$, then inspect the following statements (i) $f_x \in L^1(\nu)$ for almost $x \in X$. (ii) $f^y \in L^1(\mu)$ for almost $y \in Y$. (iii) For each $x \in X$ and $y \in Y$, define φ and ψ by $\varphi(x) = \int_Y f_x\, d\nu, \psi(y) = \int_X f^y\, d\mu$. Then the functions $\varphi \in L^1(\mu),\ \psi \in L^1(\nu)$ and $\int_X \varphi\, d\mu = \int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \psi\, d\nu$.
	SECTION C – K4 (CO3)
	Answer any TWO of the following **(2 x 12.5 = 25)**
8	Define $f: R^2 \to R^2$ by $f(x, y) = (3x + 2y + y^2 + \lvert xy\rvert, 2x + 3y + x^2 + \lvert xy\rvert)$. Verify the following: (i) f is differentiable at the origin. (ii) f is continuous at the origin. (iii) Df is invertible at the origin.
9	Determine the value of the integral $\int_0^\infty e^{-x} cos\sqrt{x}\, dx$ by applying appropriate theorem.
10	Examine the statement: Let $\{f_n\}$ be a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that $f_n \to f$ in measure.
11	Express Hahn decomposition theorem and prove it. Is the decomposition always the unique? Provide justification.
	SECTION D – K5 (CO4)
	Answer any ONE of the following **(1 x 15 = 15)**
12	State implicit function theorem and establish its proof.
13	Prove Minkowski's inequality for $p \geq 1$ and $f, g \in L^p(\mu)$. Examine the necessary conditions under which equality holds and justify.
	SECTION E – K6 (CO5)
	Answer any ONE of the following **(1 x 20 = 20)**
14	(i) Construct a non-measurable set and prove its existence. (ii) Suppose that $[\![X, \mathcal{S}]\!]$ and $[\![Y, \mathcal{T}]\!]$ be measurable spaces and $E \in \mathcal{S} \times \mathcal{T}$. Demonstrate that $E_x \in \mathcal{T}, E^y \in \mathcal{S}$ for each $x \in X$ and $y \in Y$. (15 + 5)
15	Validate the statement: Let $\{f_n\}, n \geq 1$ be a sequence of non-negative measurable functions. Then $liminf \int f_n dx \geq \int liminf f_n dx$. Use the statement to deduce Lebesgue monotone convergence theorem.

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